Applied Mathematics Letters 25 (2012) 2322–2326

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Remarks on asymptotic expansions for the gamma function

Chao-Ping Chen *, Long Lin

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454003, Henan Province, China

ARTICLE INFO

Article history:
Received 26 April 2012
Accepted 27 June 2012

Keywords:
Gamma function
Bernoulli numbers
Asymptotic expansion

ABSTRACT

We unify several asymptotic expansions for the gamma function due to Laplace, Ramanujan–Karatsuba, Gosper, Mortici, Nemes and Batir. Furthermore we present new asymptotic expansions for the gamma function.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1.1}$$

has many applications in statistical physics, probability theory and number theory. Actually it was discovered by De Moivre (1667–1754) in the form

$$n! \approx C \cdot \sqrt{n}(n/e)^n,$$

and Stirling (1692–1770) identified the constant C precisely $\sqrt{2\pi}$.

The following asymptotic formulas are well-known for the gamma function (see, for example, [1, p. 257]):

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right)\\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\cdots\right) \quad (x\to\infty) \text{ (Stirling series)}\end{aligned} \tag{1.2}$$

and

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\cdots\right)$$
$$(x\to\infty) \text{ (Laplace formula)}, \tag{1.3}$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are the n-th Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi).$$

The Laplace formula (1.3) is sometimes incorrectly called Stirling series (see [2, pp. 2–3]).

* Corresponding author.
E-mail addresses: chenchaoping@sohu.com (C.-P. Chen), linlong@hpu.edu.cn (L. Lin).

0893-9659/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2012.06.025

Stirling's formula is in fact the first approximation to the asymptotic formula (1.3). Gosper [3] replaced $\sqrt{2\pi n}$ by $\sqrt{2\pi(n+1/6)}$ in Stirling's formula to substantially improve it, to

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{6n}\right)^{1/2},$$

which was improved by Mortici [4] and Batir [5]:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{6n}+\frac{1}{72n^2}-\frac{31}{6480n^3}-\frac{139}{155520n^4}+\frac{9871}{6531840n^5}\right)^{1/2} \tag{1.4}$$

and

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{3n}+\frac{1}{18n^2}-\frac{2}{425n^3}-\frac{31}{9720n^4}\right)^{1/4}. \tag{1.5}$$

By replacing in Stirling's formula $\sqrt{2n}$ by $\left(8n^3+4n^2+n+\frac{1}{30}\right)^{1/6}$, Ramanujan [6, p. 339] presented the following approximation:

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3+4n^2+n+\frac{1}{30}\right)^{1/6}.$$

This formula has been studied in detail in [7] and [8]. Karatsuba [7, Eq. (5.5)] established the asymptotic representation of the gamma function:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\frac{79}{3360x^2}+\frac{3539}{201600x^3}-\frac{9511}{403200x^4}\right.$$
$$\left.-\frac{10051}{716800x^5}+\frac{233934691}{6386688000x^6}+\cdots\right)^{1/6} \quad (x\to\infty)$$

(x^{-6} term corrected), i.e.,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}-\frac{11}{1920x^4}+\frac{79}{26880x^5}+\frac{3539}{1612800x^6}\right.$$
$$\left.-\frac{9511}{3225600x^7}-\frac{10051}{5734400x^8}+\frac{233934691}{51093504000x^9}+\cdots\right)^{1/6} \quad (x\to\infty). \tag{1.6}$$

Moreover, the author gave a formula for successively determining the coefficients.

Nemes [9] presented a new asymptotic expansion for the gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^2}+\frac{1}{1440x^4}+\frac{239}{362880x^6}-\frac{46409}{87091200x^8}+\cdots\right)^x \quad (x\to\infty). \tag{1.7}$$

Mortici [10] published the following approximation formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}+\frac{1}{12en}\right)^n = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n^2}\right)^n. \tag{1.8}$$

Clearly, (1.7) includes (1.8) as its special case.

The asymptotic expansions (1.3)–(1.7) prompt us to ask a natural question: Does there exist an asymptotic expansion which unifies (and possibly also extends) the asymptotic expansions (1.3)–(1.7)? Our first aim in this paper is to answer this question (see Theorem 2.1).

Our second aim in this paper is to establish new asymptotic expansions for the gamma function. More precisely, let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer, our Theorem 2.2 determines the constants $a_j = a_j(\ell, r)$ ($j \in \mathbb{N}$) such that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left[1+\ln\left(1+\sum_{j=1}^{\infty}\frac{a_j}{x^j}\right)\right]^{x^\ell/r} \quad (x\to\infty).$$

2. Main results

Theorem 2.1 unifies the asymptotic expansions (1.3)–(1.7).

Theorem 2.1. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\sum_{j=1}^{\infty}\frac{b_j}{x^j}\right)^{x^\ell/r} \quad (x\to\infty), \tag{2.9}$$

where the coefficients $b_j = b_j(\ell, r)$ $(j \in \mathbb{N})$ are given by

$$b_j = b_j(\ell, r) = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{B_2}{1\cdot 2}\right)^{k_1}\left(\frac{B_3}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{B_{j+1}}{j(j+1)}\right)^{k_j}, \tag{2.10}$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (2+\ell)k_2 + \cdots + (j+\ell)k_j = j.$$

Proof. It is easy to see from (1.2) that (2.9) holds true for some real numbers $b_1, \ldots, b_m$. What matters is to determine the $b_1, \ldots, b_m$ explicitly. To do this, we first express (2.9) as follows:

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}\right)^{r/x^\ell} = 1+\sum_{j=1}^{m}\frac{b_j}{x^j} + O(x^{-m-1}) \quad (x\to\infty). \tag{2.11}$$

Write (1.2) as

$$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x} = \exp\left(\sum_{k=1}^{m}\frac{B_{k+1}}{k(k+1)x^k} + \mathcal{R}_m(x)\right) \quad (x\to\infty),$$

where $\mathcal{R}_m(x) = O(x^{-m-1})$. Further, we have

$$\begin{aligned}\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}\right)^{r/x^\ell} &= e^{r\mathcal{R}(x)/x^\ell}\exp\left(\sum_{k=1}^{m}\frac{rB_{k+1}}{k(k+1)x^{k+\ell}}\right)\\ &= e^{r\mathcal{R}(x)/x^\ell}\prod_{k=1}^{m}\left[1+\left(\frac{rB_{k+1}}{k(k+1)x^{k+\ell}}\right)+\frac{1}{2!}\left(\frac{rB_{k+1}}{k(k+1)x^{k+\ell}}\right)^2+\cdots\right]\\ &= e^{r\mathcal{R}(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}\left(\frac{rB_2}{1\cdot 2}\right)^{k_1}\left(\frac{rB_3}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{rB_{m+1}}{m(m+1)}\right)^{k_m}\\ &\quad\cdot\frac{1}{x^{(1+\ell)k_1+(2+\ell)k_2+\cdots+(m+\ell)k_m}}.\end{aligned} \tag{2.12}$$

Equating the coefficients by the equal powers of x in (2.11) and (2.12), we see that

$$b_j = \sum_{(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j}\frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{B_2}{1\cdot 2}\right)^{k_1}\left(\frac{B_3}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{B_{j+1}}{j(j+1)}\right)^{k_j}.$$

This completes the proof of Theorem 2.1. □

Remark 2.1. (i) We find that special cases of (2.9) when $r = 1, 2, 4, 6$ and $\ell = 0$ yield immediately the formulas (1.3)–(1.6), respectively. Setting $(r, \ell) = (1, 1)$ in (2.9), we obtain the asymptotic expansion (1.7). Laplace formula (1.3) can be rewritten as

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\sum_{\ell=1}^{\infty}\frac{c_\ell}{x^\ell}\right) \quad (x\to\infty;\ \ell\in\mathbb{N}), \tag{2.13}$$

where the coefficients c_ℓ $(\ell \in \mathbb{N})$ are given by

$$c_\ell = \sum_{k_1+2k_2+\cdots+\ell k_\ell=\ell}\frac{1}{k_1!k_2!\cdots k_\ell!}\left(\frac{B_2}{1\cdot 2}\right)^{k_1}\left(\frac{B_3}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{B_{\ell+1}}{\ell(\ell+1)}\right)^{k_\ell}. \tag{2.14}$$

(ii) Here, from (2.9), we give two new asymptotic expansions:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 - \frac{1}{12x} + \frac{1}{288x^2} + \frac{139}{51840x^3} - \frac{571}{2488320x^4} - \frac{163879}{209018880x^5} + \cdots\right)^{-1} \quad (x \to \infty) \tag{2.15}$$

and

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 - \frac{1}{6x} + \frac{1}{72x^2} + \frac{31}{6480x^3} - \frac{139}{155520x^4} - \frac{9871}{6531840x^5} + \cdots\right)^{-1/2} \quad (x \to \infty). \tag{2.16}$$

In 2008, Batir [11] presented the following approximation to $n!$:

$$n! \sim \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 - \frac{1}{6n}\right)^{-1/2}. \tag{2.17}$$

Clearly, (2.16) includes (2.17) as its special case.

Theorem 2.2 presents new asymptotic expansions for the gamma function.

Theorem 2.2. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The following asymptotic expansion holds:*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left[1 + \ln\left(1 + \sum_{j=1}^{\infty}\frac{a_j}{x^j}\right)\right]^{x^\ell/r} \quad (x \to \infty), \tag{2.18}$$

where the coefficients $a_j = a_j(\ell, r)$ $(j \in \mathbb{N})$ are given by

$$a_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{1}{k_1!k_2!\cdots k_j!} b_1^{k_1} b_2^{k_2}\cdots b_j^{k_j}, \tag{2.19}$$

and b_j $(j \in \mathbb{N})$ are determined in (2.10).

Proof. A similar argument as in the proof of Theorem 2.1 will establish the result in Theorem 2.2. For completeness, we repeat to prove Theorem 2.2. To determine some real numbers $a_1, \ldots, a_m$, we first express (2.18) as follows:

$$e^{\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)^{r/x^\ell} - 1} = 1 + \sum_{j=1}^{m}\frac{a_j}{x^j} + O\left(x^{-m-1}\right) \quad (x \to \infty;\ m \in \mathbb{N}). \tag{2.20}$$

Write (2.9) as

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}\right)^{r/x^\ell} - 1 = \sum_{k=1}^{m}\frac{b_k}{x^k} + R(x) \quad (x \to \infty),$$

where $R(x) = O(x^{-m-1})$. Further, we have

$$\begin{aligned} e^{\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)^{r/x^\ell} - 1} &= e^{R(x)} e^{\sum_{k=1}^{m}\frac{b_k}{x^k}} \\ &= e^{R(x)} \prod_{k=1}^{m}\left[1 + \left(\frac{b_k}{x^k}\right) + \frac{1}{2!}\left(\frac{b_k}{x^k}\right)^2 + \cdots\right] \\ &= e^{R(x)} \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!} b_1^{k_1} b_2^{k_2}\cdots b_m^{k_m} \cdot \frac{1}{x^{k_1+2k_2+\cdots+mk_m}}. \end{aligned} \tag{2.21}$$

Equating the coefficients by the equal powers of x in (2.20) and (2.21), we see that

$$a_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{1}{k_1!k_2!\cdots k_j!} b_1^{k_1} b_2^{k_2}\cdots b_j^{k_j},$$

where b_j $(j \in \mathbb{N})$ are determined in (2.10). This completes the proof of Theorem 2.2. □

Setting $(r, \ell) = (1, 0)$ and $(r, \ell) = (1, 1)$ in (2.18), respectively, we obtain two explicit expressions:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left[1 + \ln\left(1 + \frac{1}{12x} + \frac{1}{144x^2} - \frac{119}{51840x^3} - \frac{359}{829440x^4} + \cdots\right)\right] \quad (x \to \infty) \tag{2.22}$$

and

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left[1 + \ln\left(1 + \frac{1}{12x^2} + \frac{1}{240x^4} - \frac{59}{72576x^6} - \frac{1963}{4147200x^8} + \cdots\right)\right]^x \quad (x \to \infty). \tag{2.23}$$

References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, in: Applied Mathematics Series, vol. 55, National Bureau of Standards, Washington, DC, 1972. Ninth printing.
[2] E.T. Copson, Asymptotic Expansions, Cambridge University Press, 1965.
[3] R.W. Gosper, Decision procedure for indefinite hypergeometric summation, Proc. Natl. Acad. Sci. USA 75 (1978) 40–42.
[4] C. Mortici, Sharp inequalities related to Gosper's formula, C. R. Math. Acad. Sci. Paris 348 (2010) 137–140.
[5] N. Batir, Very accurate approximations for the factorial function, J. Math. Inequal. 4 (2010) 335–344.
[6] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Springer, Berlin, 1988.
[7] E.A. Karatsuba, On the asymptotic representation of the Euler gamma function by Ramanujan, J. Comput. Appl. Math. 135 (2001) 225–240.
[8] H. Alzer, On Ramanujan's double inequality for the gamma function, Bull. London Math. Soc. 35 (2003) 601–607.
[9] G. Nemes, New asymptotic expansion for the Gamma function, Arch. Math. (Basel) 95 (2010) 161–169.
[10] C. Mortici, A substantial improvement of the Stirling formula, Appl. Math. Lett. 24 (2011) 1351–1354.
[11] N. Batir, Sharp inequalities for factorial *n*, Proyecciones 27 (2008) 97–102.